Sect 4.2 - Solving Systems of Equations by Substitution

Concept #2 Solving System of Linear Equations in Two Variables by Substitution.

In this section, we will solve one equation for one variable and substitute that expression for that variable in the other equation. This will give us one linear equation with one variable. We solve that equation and that will yield the value for one of the coordinates of our solution. We take that answer, substitute it into one of the two original equations and solve to find the value for the other coordinate of our solution. We will use the last example from the previous section as our first example.

Solve by substitution:

Ex. 1 $6x - 2y = 4$

$y = -\frac{1}{2}x$

Solution:

Equation #2 is already solved for y, so y and $-\frac{1}{2}x$ are the same thing in this problem. In equation #1, replace y by $\mathbf{-\frac{1}{2}x}$:

$6x - 2y = 4$

$6x - 2\left(-\frac{1}{2}\mathbf{x}\right) = 4$ (solve for x)

$6x + x = 4$

$7x = 4$

$x = \frac{4}{7}$ Since $y = -\frac{1}{2}x$, then $y = -\frac{1}{2}\left(\frac{4}{7}\right) = -\frac{2}{7}$.

So, the solution is $\left(\frac{4}{7}, -\frac{2}{7}\right)$.

Ex. 2 $x + \frac{1}{4}y = \frac{5}{4}$

$2x - 3y = 13$

Solution:

It is easiest to solve equation #1 for x:

$x + \frac{1}{4}y = \frac{5}{4}$ $\Rightarrow$ $x = -\frac{1}{4}y + \frac{5}{4}$

Now, replace x in equation #2 by $\mathbf{-\frac{1}{4}y + \frac{5}{4}}$:

$2x - 3y = 13$

$2\left(-\frac{1}{4}\mathbf{y} + \frac{5}{4}\right) - 3y = 13$ (distribute)

$-\frac{1}{2}y + \frac{5}{2} - 3y = 13$ (multiply both sides by 2)

$-y + 5 - 6y = 26$ (combine like terms)

$-7y + 5 = 26$ (solve)

$-7y = 21$

$y = -3$ Since $x = -\frac{1}{4}y + \frac{5}{4}$, then $x = -\frac{1}{4}(-3) + \frac{5}{4}$

$= -\frac{1}{4}(-\frac{3}{1}) + \frac{5}{4} = \frac{3}{4} + \frac{5}{4} = \frac{8}{4} = 2.$

So, the solution is (2, – 3).

Ex. 3 $\frac{1}{4}x - 2y = 1$

$x - 8y = 4$

Solution:

Equation #2 is easiest to solve for x:

$x - 8y = 4 \quad \Rightarrow \quad x = 8y + 4$

Now, replace x in equation #1 by **8y + 4**:

$\frac{1}{4}x - 2y = 1$

$\frac{1}{4}(\mathbf{8y + 4}) - 2y = 1$ (distribute)

$2y + 1 - 2y = 1$ (combine like terms)

$1 = 1$

Back in chapter 2, when we obtained an equation like this, we said the solution was all real numbers. In the context of solving systems of equations, it does not mean all real numbers, but rather the lines are the same line. Hence, every point on the line is a solution. Thus, the solution is $\{ (x, y) \mid x - 8y = 4\}$.

Ex. 4 $4x + 2y = -3$

$10x + 5y = 2$

Solution:

Solve equation #1 for y:

$4x + 2y = -3 \quad \Rightarrow \quad 2y = -4x - 3 \quad \Rightarrow \quad y = -2x - 1.5$

In equation #2, replace y with **– 2x – 1.5**:

$10x + 5y = 2$

$10x + 5(\mathbf{-2x - 1.5}) = 2$ (distribute)
$10x - 10x - 7.5 = 2$ (combine like terms)
$-7.5 = 2$

Back in chapter 2, when we obtained an equation like this, we said there was no solution. In the context of solving systems of equations, it means that the lines are parallel. Hence, the system has no solution.

Ex. 5 $\frac{2}{3}x - \frac{1}{6}y = \frac{5}{6}$
$-\frac{1}{5}x + \frac{1}{3}y = \frac{3}{5}$

Solution:
It is easiest to clear fractions first. The L.C.D. of equation #1 is 6 and the L.C.D. of equation #2 is 15.

1) $\frac{2}{3}x - \frac{1}{6}y = \frac{5}{6}$ (multiply both sides by 6)
$6\left(\frac{2}{3}x\right) - 6\left(\frac{1}{6}y\right) = 6\left(\frac{5}{6}\right) \Rightarrow 4x - y = 5$

2) $-\frac{1}{5}x + \frac{1}{3}y = \frac{3}{5}$ (multiply both sides by 15)
$15\left(-\frac{1}{5}x\right) + 15\left(\frac{1}{3}y\right) = 15\left(\frac{3}{5}\right) \Rightarrow -3x + 5y = 9$

Now, solve equation #1 for y:
$4x - y = 5 \Rightarrow -y = -4x + 5 \Rightarrow y = 4x - 5$

Finally, replace y in equation #2 with **4x – 5**:
$-3x + 5(\mathbf{4x - 5}) = 9$ (distribute)
$-3x + 20x - 25 = 9$ (combine like terms)
$17x - 25 = 9$ (solve)
$17x = 34$
$x = 2$ Since $y = 4x - 5$, then $y = 4(2) - 5 = 8 - 5 = 3$.

The solution is (2, 3).

Ex. 6 $x = 4$
$y = -5$

Solution:
Since $x = 4$ and $y = -5$, then the solution is $(4, -5)$.

Concept #2 Solutions to Systems of Linear Equations: A Summary

Case 1: The lines intersect at one point yielding one unique solution.

Case 2: The lines are parallel and do not intersect. This is determined when a false statement is obtained in the solution (i.e., 5 = – 3).

Case 3: The lines are the same line and intersect at all the points on the line. This is determined when a true statement is obtained in the solution (i.e., – 4 = – 4).

Concept #3 Applications of the Substitution Method

Write the following as a system of two linear equations and solve:

Ex. 7 Two positive numbers have a difference of 12. The larger number is four less than three times the smaller number. Find the numbers.

Solution:

Let L = the larger positive number
s = the smaller positive number
"Two positive numbers have a difference of 12": L – s = 12
"The larger number is four less than three times the smaller number": L = 3s – 4

So, our system of equations is

1) L – s = 12
2) **L = 3s – 4**

In equation #1, replace **L** by **3s – 4**:

L – s = 12
(**3s – 4**) – s = 12
2s – 4 = 12
2s = 16
s = 8

Since **L = 3s – 4**, then

L = 3(8) – 4 = 20

So, the numbers are 8 and 20.

Ex. 8 Two angles are complementary. The measure of one angle is 18˚ more than twice the measure of the other angle, Find the measures of the angles.

Solution:

Let A = the measure of one angle
B = the measure of the other angle

"Two angles are complementary": The sum of the measures of the angles is 90˚. Hence, A + B = 90

"The measure of one angle is 18˚ more than twice the measure of the other angle: one angle is 18 more than twice the other

A = 2B + 18

Thus, our system of equations is:

1) A + B = 90

2) **A = 2B + 18**

In equation #1, replace **A** by **2B + 18**:

(**2B + 18**) + B = 90

3B + 18 = 90

3B = 72

B = 24˚ and A = 2(24) + 18 = 48 + 18 = 66˚

So, the angles are 24˚ and 66˚.

Ex. 8 If the measure of the third angle of an isosceles triangle is six degrees less than three times the measure of one of the two equal base angles, find the measure of each angle.

Solution:

Let E = the measure of one of the two equal base angles

T = the measure of the third angle

Recall that the sum of the measures of the angles of a triangle is 180˚. That is: 1st angle + 2nd angle + 3rd angle = 180

E + E + T = 180

2E + T = 180

The measure of the third angle of an isosceles triangle is six degrees less than three times the measure of one of the two equal base angles: 3rd angle = 3•equal angle – 6

T = 3E – 6

Thus, our system is:

1) 2E + T = 180

2) **T = 3E – 6**

In equation #1, replace **T** by **3E – 6**:

2E + (**3E – 6**) = 180

5E – 6 = 180

5E = 186

E = 37.2 and T = 3E – 6 = 3(37.2) – 6 = 105.6.

So, the angles are 37.2˚, 37.2˚, and 105.6˚.